STABILITY OF VIBRATIONS OF FINITE AMPLITUDE IN AN ELECTRON–ION RING

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The article discusses transverse vibrations of finite amplitude in an electron–ion ring. Far from the region of linear resonances, an equation is obtained for slowly varying amplitudes, and the conditions are found for the excitation of instability of the "negative pressure" type. Near the lower boundary of the region of linear instability, the conditions are found under which nonlinearity breaks down the stability of the vibrations with finite amplitudes.

1. In [1] a study was made of the question of the stability of an electron–ion ring with respect to transverse vibrations (instability of the "spiral" type in the approximation of the linearization of the polarization force, with a relative shift of the centers of gravity of the bundles. It is found that there exists a region of wave numbers ($k_- < k < k_+$) for which there is instability. The appearance of transverse instability is connected with the presence of resonances at the frequencies of the electron–ion vibrations (near k_+) and at Doppler frequencies of the vibrations of the electrons in a focused magnetic field (near k_-). The nonlinear stage of the development of the vibrations shows that, near the upper boundary of the region of instability (k_+), the nonlinear polarization force stabilizes the instability at amplitudes which are small in comparison with the transverse radius of the bundle [2]. In the case where there is no external focusing of the electrons, in [2] an exact solution was obtained in the form of a nonlinear stationary wave, moving along the bundle. With nonlinearity in an electron–ion annulus, there is the possibility of vibrations with finite amplitudes, both near the region of linear instability and far from it.

In the present article, equations are obtained which describe vibrations of this type, and a study is made of the question of their stability.

The models chosen were two rigid bundles of electrons and ions, whose particle density in a transverse cross section is distributed in accordance with the Gaussian law

$$n_{e,i}(r_1) = n_{e,i}^{(0)} \exp(-r_\perp^2 / a_0^2) \tag{1.1}$$

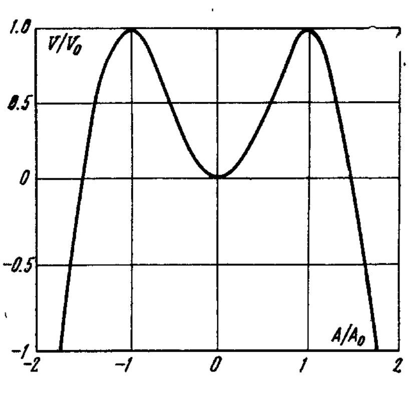

Fig. 1

where a_0 is a constant characterizing the radius of the bundles; $r_\perp$ is the modulus of the radius vector of a particle in the plane of the transverse cross section of the bundle. Measuring all the space and time quantities in the units a_0 and ω_0, respectively. ($\omega_0^2 = \pi e^2 n_c^{(0)}/M_i$ is the frequency of the vibrations of the ions in the field of the electrons), we write the system of equations for the transverse displacements of the centers of gravity of the bundles [1, 2],

$$d^2x/dt^2 + \lambda^2 x = -\delta(x-y)[1 - {}^1/_4(y-x)^2], \quad d^2y/dt^2 = (x-y)[1 - {}^1/_4(y-x)^2]$$

$$d/dt \equiv \partial/\partial t + v\partial/\partial z, \quad \delta = n_i^{(0)} M_i/\gamma n_e^{(0)} m_e, \quad \gamma = (1 - v^2/c^2)^{-1/2} \tag{1.2}$$

Here x(y) is the shift of the electrons (ions); λ is the frequency of the vibrations in an external focused field; v is the velocity of the electrons.

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The polarization force in (1.2), arising with a transverse shift of the centers of gravity of the bundles, is calculated for the distribution (1.1) under the assumption of a small degree of nonlinearity ($x, y \ll 1$).

With the satisfaction of the latter condition, we use the method of "extension," developed in [3-5]. We seek the solution of (1.2) in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \sum_{n=1}^{\infty} \varepsilon^n \sum_{l=-\infty}^{\infty} \begin{pmatrix} x_l^{(n)}(\xi, \tau) \\ y_l^{(n)}(\xi, \tau) \end{pmatrix} e^{il(kz-\omega t)} \tag{1.3}$$

where $\varepsilon \ll 1$, ξ and τ vary slowly,

$$\xi = \varepsilon (z - c_0 t), \quad c_0 = \partial\omega/\partial k, \quad \tau = \varepsilon^2 t \tag{1.4}$$

With an accuracy up to ε, from (1.2) we obtain the dispersion equation of the linear theory [1],

$$F(\omega) = \omega^{-2} + \delta/[(\omega - kv)^2 - \lambda^2] = 1$$

Taking into account that

$$c_0 = v/[1 + \delta\omega/\alpha^2(\omega - kv)]^{-1} \tag{1.5}$$

with an accuracy to ε^2 we obtain

$$x_1^{(2)} - \alpha y_1^{(2)} = 2ic_0\omega\,\partial\Phi/\partial\xi, \quad \Phi(\xi, \tau) \equiv y_1^{(1)}(\xi, \tau), \quad \alpha = 1 - \omega^2 \tag{1.6}$$

Taking account of (1.5), (1.6), in the order ε^3, we obtain an equation for $\Phi(\xi, \tau)$,

$$\partial\Phi/\partial\tau + i\nu\partial^2\Phi/\partial\xi^2 = iq|\Phi|^2\Phi \tag{1.7}$$

$$\nu = \frac{1}{2}\left[\frac{4c_0^2\omega^2}{\alpha} + \frac{\alpha^2(c_0 - v)^2}{\delta} + c_0^2\right]\left[\frac{\alpha^2(\omega - kv)}{\delta} + \omega\right]^{-1/2}$$

$$q = {}^3/_4(1 - \alpha)^4[\alpha^2(\omega - kv)/\delta^{-1} + \omega]^{-1/2} \tag{1.8}$$

An equation of the type (1.7) was investigated, for example, in [4-6]. We shall show the conditions under which instability of the negative-pressure type is possible. We seek the solution of (1.7) in the form

$$\Phi(\xi, \tau) = \sqrt{\rho}\exp\left(-\frac{i}{2\nu}\int\sigma d\xi\right) \tag{1.9}$$

Substituting (1.9) into (1.7), and separating the real and imaginary parts, we obtain a system of equations for determining ρ and σ,

$$\frac{\partial\rho}{\partial\tau} + \frac{\partial}{\partial\xi}(\sigma\rho) = 0, \quad \frac{\partial\sigma}{\partial\tau} + \sigma\frac{\partial\sigma}{\partial\xi} = -2\nu q\frac{\partial\rho}{\partial\xi} + 2\nu^2\frac{\partial}{\partial\xi}\frac{1}{\sqrt{\rho}}\frac{\partial^2}{\partial\xi^2}\sqrt{\rho} \tag{1.10}$$

The last term in the second equation of (1.10) is small under the condition

$$q\Delta^2/k^2\nu \gg 1 \tag{1.11}$$

where $\Delta = |x-y|_{max}$ is the amplitude of the relative shift of the centers of gravity of the bundles. The quantity $q\nu\rho^2$ in (1.10) plays the role of the pressure, if ρ and σ are regarded as the hydrodynamic density and velocity. With $q\nu < 0$, the "pressure" becomes negative, and the initial perturbations rise exponentially. The condition $q\nu < 0$ gives

$$4c_0^2\omega^2/\alpha + \alpha^2(c_0 - v)^2/\delta + c_0^2 < 0 \tag{1.12}$$

Let us examine the ionic branch of the vibrations

$$\omega \approx 1 + \delta/2k^2v^2, \quad \alpha \approx -\delta/k^2v^2, \quad c_0 \approx -\delta/k^3v^2 \tag{1.13}$$

Taking account of (1.13), from (1.11), (1.12) we obtain the conditions for the appearance of instability of the negative-pressure type,

$$\delta^{-1}(kv\Delta/\omega_0)^2 \gg 1, \quad \delta(\omega_0/kv)^3 \ll 1 \tag{1.14}$$

With small values of δ (the case of small condensation), condition (1.14) is satisfied well even with relatively small values of Δ.

2. Let us consider the stability of vibrations of finite amplitude near the lower boundary of the region of linear instability (k_-). Using a method analogous to that of Sec. 1, we obtain an equation for $\Phi(\xi, \tau)$ near the boundary of the region of linear instability ($\partial F/\partial\omega = 0$),

$$i\partial\Phi/\partial\xi = a\partial^2\Phi/\partial\tau^2 - (b\,|\,\Phi\,|^2 + c\Phi),\quad \tau = \varepsilon t,\quad \xi = \varepsilon^2 z \tag{2.1}$$

$$a = -\tfrac{1}{2}\,[4\omega^2\delta/\alpha + \alpha^2 k v_0/\omega](v_0\omega\delta)^{-1}$$

$$b = -\tfrac{3}{4}\,(1-\alpha)^4(v_0\omega)^{-1},\quad c = (\mu/\,|\,\mu\,|)k,\quad \mu = (v - v_0)/v_0,\quad |\,\mu\,| \ll 1 \tag{2.2}$$

where v_0 is the velocity at the boundary of the region of instability. We select a vibration with the wave vector $k < k_-$, i.e., in a linear approximation the vibration is stable ($\mu > 0$), and we consider the case $\sqrt{\delta/\lambda} = \varkappa \ll 1$; $\lambda \gg 1$, $(\lambda \sim 1)$.

From (2.2) we obtain

$$a_- \approx -2k/(1+\lambda)\varkappa,\quad b_- \approx -3k/4\,(1+\lambda) \tag{2.3}$$

We seek the solution of (2.1) in the form

$$\Phi\,(\xi,\ \tau) = A\,(\tau)\exp\,(-i\psi\,(\tau)) \tag{2.4}$$

We substitute (2.4) into (2.1) and separate the real and imaginary parts; we then integrate the equation for the amplitude $A(\tau)$ once.

We obtain

$$\tfrac{1}{2}\,(dA/d\tau)^2 + V\,(A) = H = \text{const} \tag{2.5}$$

$$V\,(A) = -M^2/A^2 - (b/4a)\,A^4 - (c/2a)\,A^2,\quad M = A^2 d\psi/d\tau = \text{const} \tag{2.6}$$

Let us consider vibrations with a zero "moment," $M = 0$. The solution of Eq. (2.5) has the form

$$A\,(\tau) = A_0\sqrt{1-\beta^2}\,\mathrm{Sn}\,[\sqrt{\tfrac{1}{2}\,|\,c\,/\,a\,|\,(1+\beta)}\;\tau;\ \sqrt{(1-\beta)\,/\,(1+\beta)}] \tag{2.7}$$

$$\beta = \sqrt{1 - H\,/\,V_0},\qquad A_0 = \sqrt{|\,c\,/\,b\,|},\qquad V_0 = c^2\,/\,4\,|\,ab\,| \tag{2.8}$$

Figure 1 illustrates the potential curve (2.6). The values of A_0 and V_0 on Fig. 1 are calculated using (2.8). In the case (2.3), an evaluation of the value of the maximal amplitude $\Delta = |x-y|_{max}$ corresponding to A_0 in (2.7), (2.8) gives

$$\Delta \approx 4\varepsilon\sqrt{(1+\lambda)\,/\,3} \tag{2.9}$$

The quantity ε corresponds to a shift with respect to k from the lower boundary of the region of instability from the side of small values of k, into the region of linear stability ($\varepsilon^2 \sim |\,\mu\,|$): $k_- - k \approx \varepsilon^2 k_-$. Therefore, assigning small values for the quantity ε, we obtain small critical amplitudes in expression (2.9), at which there is a breakdown of the stability of vibrations of finite amplitude near the lower boundary of the region of instability.

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LITERATURE CITED

1. B. V. Chirikov, "The stability of a partially compensated electron beam," At. Énerg., 19, No. 3 (1965).
2. N. Yu. Kazarinov, A. B. Kuznetsov, E. A. Perel'shtein, S. V. Rubin, and V. F. Shevtsova, "Nonlinear phenomena in instability of the "spiral" type of an electron–ion ring," Preprint P9-6284, Dubna (1972).
3. Y. Midzuno and T. Watanabe, "Nonlinear behavior of waves in a plasma in the vicinity of a marginally stable state," J. Phys. Soc. Jap., 28, No. 3 (1970).
4. T. Taniuti and N. Yajima, "Perturbation method for a nonlinear wave modulation. I," J. Math. Phys., 10, No. 8 (1968).
5. N. Asano, T. Taniuti, and N. Yajima, "Perturbation method for a nonlinear wave modulation. II," J. Math. Phys., 10, No. 11 (1968).
6. V. E. Zakharov, "The stability of periodic waves of finite amplitude on the surface of a deep liquid," Zh. Prikl. Mekhan. Tekh. Fiz., No. 2 (1968).